

CODIMENSION GROWTH OF A VARIETY OF NOVIKOV ALGEBRAS

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ABSTRACT. An algebra with identities $a \circ (b \circ c - c \circ b) = (a \circ b) \circ c - (a \circ c) \circ b$ and $a \circ (b \circ c) = b \circ (a \circ c)$ is called Novikov. We construct free Novikov base in terms of Young diagrams. We show that codimensions exponent for a variety of Novikov algebras exists and is equal 4.

1. INTRODUCTION

A variety of algebras is a class of algebras satisfying some polynomial identities. One of important parameters of varieties is so-called codimension growth. If \mathcal{V} is a variety and $N_n(\mathcal{V})$ is a polylinear part of its free algebra generated by n elements, then $c_n(\mathcal{V}) = \dim N_n(\mathcal{V})$ is called n -th codimension of \mathcal{V} . Codimension growth is defined by a sequence of codimensions c_1, c_2, c_3, \dots . *Codimension exponent* is defined as

$$Exp(\mathcal{V}) = \lim c_n(\mathcal{V})^{1/n}.$$

Appear natural questions whether this exponent exists and whether it is an integer. In associative case these questions are well studied. It was proved that $Exp(\mathcal{V})$ exists and it is an integer for any proper variety of associative algebras ([5]). Constructions of free bases for Lie algebras are well known (about Hall-Lyndon- Shirshov bases see, for example [7]).

In our paper we consider class of non-associative algebras. An algebra $A = (A, \circ)$ is called *Novikov* ([1], [6], [4]), if it satisfies the following identities

$$\begin{aligned} (a, b, c) &= (a, c, b) \\ a \circ (b \circ c) &= b \circ (a \circ c), \end{aligned}$$

for any $a, b, c \in A$. Here

$$(a, b, c) = a \circ (b \circ c) - (a \circ b) \circ c$$

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is an associator. Novikov algebras are Lie-admissible.

Example. $A = \mathbf{C}[x]$ under multiplication $a \circ b = \partial(a)b$ is Novikov.

Let Nov be a variety of Novikov algebras and N_n be a polylinear part of free Novikov algebra generated by n elements. Let

$$Exp(Nov) = \lim_{n \rightarrow \infty} (\dim N_n)^{1/n}$$

be codimensions growth of Novikov variety.

In [3] was proved that Novikov operad is not Koszul. Now we give construction of free Novikov base in terms of Young diagrams and use this base to calculation of generating function and codimension growth. Our main result is

Theorem 1.1. *Codimensions sequence for Novikov variety is given by*

$$\dim N_n = \binom{2n-2}{n-1}.$$

Codimension exponent of Novikov variety exists and

$$Exp(Nov) = 4.$$

Generating function of codimensions sequence $\sum_{i \geq 0} N_i x^i$ is equal to $x(1-4x)^{-1/2}$.

2. FREE BASE FOR NOVIKOV ALGEBRAS

In [2] are given constructions of a base of free Novikov algebra in terms of r -elements and in terms of rooted trees. In this section we give construction of free base in terms of Young diagrams.

Recall that Young diagram is a set of boxes (we denote them by bullets) with non-increasing numbers of boxes in each row. Rows and columns are numerated from top to bottom and from left to right. Let k be the number of rows and r_i be the number of boxes in i -th row. The total number of boxes, $r_1 + \dots + r_k$, is called *degree* of Young diagram.

To construct Novikov diagram, we need to complement Young diagram by one box, we call it as "a nose". Namely, we need to add the first row by one more box,

$$\begin{array}{ccccccc} \bullet & \dots & \bullet & \bullet & \bullet & & \bullet & \dots & \bullet & \bullet & \bullet & \circ \\ \bullet & \dots & \bullet & \bullet & & & \bullet & \dots & \bullet & \bullet & & \\ \vdots & \dots & \vdots & \vdots & & & \vdots & \dots & \vdots & \vdots & & \\ \bullet & \dots & \bullet & & & & \bullet & \dots & \bullet & & & \end{array} \mapsto$$

The number of boxes in Novikov diagram is called its *degree*. So, difference between degrees of Novikov diagram and corresponding Young diagram is equal to 1.

Let us given an alphabet (ordered set) Ω . To construct Novikov tableau on Ω we need to feel Novikov diagrams by elements of Ω . Denote by $a_{i,j}$ an element of Ω in the box (i, j) , that is a cross of i -th row by j -th column. The feeling rule is the following

- $a_{i,1} \geq a_{i+1,1}$, if $r_i = r_{i+1}$, $i = 1, 2, \dots, k-1$.
- the sequence $a_{k,2} \cdots a_{k,r_k} a_{k-1,2} \cdots a_{k-1,r_{k-1}} \cdots a_{1,2} \cdots a_{1,r_1} a_{1,r_1+1}$ is non-decreasing.

In particular, all boxes beginning from the second place in each row are labeled by non-decreasing elements of alphabet. Denote by R_n a set of Novikov tableaux labeled by Ω with $n = r_k + \cdots + r_1 + 1$ boxes.

Let $F(\Omega)$ be free Novikov algebra generated by Ω . Let $F_n(\Omega)$ be its subspace generated by basic elements of degree n . Correspond to any Novikov tableaux

$$\begin{array}{ccccccc} a_{1,1} & \cdots & \cdots & a_{1,r_1-1} & a_{1,r_1} & a_{1,r_1+1} \\ a_{2,1} & \cdots & a_{2,r_2-1} & a_{2,r_2} & & \\ \vdots & \cdots & \vdots & \vdots & & \\ a_{k,1} & \cdots & a_{k,r_k} & & & \end{array}$$

an element

$$X = X_k \circ (X_{k-1} \circ (\cdots \circ (X_2 \circ X_1) \cdots)),$$

(right-normed bracketing) where

$$X_i = (\cdots ((a_{i,1} \circ a_{i,2}) \circ a_{i,3}) \cdots \circ a_{i,r_i-1}) \circ a_{i,r_i}, \quad 1 < i \leq k,$$

$$X_1 = (\cdots ((a_{1,1} \circ a_{1,2}) \circ a_{1,3}) \cdots \circ a_{1,r_1}) \circ a_{1,r_1+1}$$

(left-normed bracketing). All base elements of free Novikov algebra $F(\Omega)$ are obtained by this way. In particular, $\dim F_n(\Omega) = |R_n|$.

As an example let us construct base elements of polylinear part N_4 of free Novikov algebra generated by 4 elements a, b, c, d .

Young diagrams of degree 3:

$$\begin{array}{ccccccc} \bullet & & \bullet & \bullet & & \bullet & \bullet & \bullet \\ \bullet & & \bullet & & & & & \\ \bullet & & & & & & & \end{array}$$

Novikov diagrams of degree 4:

$$\begin{array}{ccccccc} \bullet & \circ & & \bullet & \bullet & \circ & & \bullet & \bullet & \bullet & \circ \\ \bullet & & & \bullet & & & & & & & \\ \bullet & & & & & & & & & & \end{array}$$

Novikov tableaux of degree 4 generated by elements a, b, c, d :

$$\begin{array}{cccc} c & d & d & c & d & b & d & a \\ b & & b & & c & & c & \\ a & & a & & a & & b & \end{array}$$

$$\begin{array}{ccc} b & c & d & c & b & d & d & b & c \\ a & & & a & & & a & & \end{array}$$

$$\begin{array}{ccc} a & c & d & c & a & d & d & a & c \\ b & & & b & & & b & & \end{array}$$

$$\begin{array}{ccc} a & b & d & b & a & d & d & a & b \\ c & & & c & & & c & & \end{array}$$

$$\begin{array}{ccc} a & b & c & b & a & c & c & a & b \\ d & & & d & & & d & & \end{array}$$

$$a \ b \ c \ d \quad b \ a \ c \ d \quad c \ a \ b \ d \quad d \ a \ b \ c$$

So, polylinear part of free Novikov algebra in degree 4 is 20-dimensional and the following elements form base

$$a \circ (b \circ (c \circ d)), a \circ (b \circ (d \circ c)), a \circ (c \circ (d \circ b)), b \circ (c \circ (d \circ a)),$$

$$a \circ ((b \circ c) \circ d), a \circ ((c \circ b) \circ d), a \circ ((d \circ b) \circ c),$$

$$b \circ ((a \circ c) \circ d), b \circ ((c \circ a) \circ d), b \circ ((d \circ a) \circ c),$$

$$c \circ ((a \circ b) \circ d), c \circ ((b \circ a) \circ d), c \circ ((d \circ a) \circ b),$$

$$d \circ ((a \circ b) \circ c), d \circ ((b \circ a) \circ c), d \circ ((c \circ a) \circ b),$$

$$((a \circ b) \circ c) \circ d, ((b \circ a) \circ c) \circ d, ((c \circ a) \circ b) \circ d, ((d \circ a) \circ b) \circ c.$$

3. CODIMENSIONS GROWTH OF NOVIKOV VARIETY

Let $\lambda = 1^{m_1(\lambda)} 2^{m_2(\lambda)} \dots$ be a partition of $n - 1$, i.e.,

$$|\lambda| = \sum_{i \geq 1} i m_i(\lambda) = n - 1.$$

Let

$$m(\lambda) = \sum_{i \geq 1} m_i(\lambda).$$

For m, m_1, m_2, \dots, m_n , such that $m = m_1 + m_2 + \dots + m_n$ let

$$\binom{m}{m_1, m_2, \dots, m_n} = \frac{m!}{m_1! m_2! \dots m_n!}$$

be a multinomial coefficient.

Lemma 3.1.

$$\sum_{|\lambda|=n-1} \binom{m(\lambda)}{m_1(\lambda), m_2(\lambda), \dots} \binom{n}{m(\lambda)} = \binom{2(n-1)}{n-1}$$

Proof. By Vandermonde convolution relation [8], chapter 1.3, formulae (3a),

$$\binom{n+p}{m} = \sum_{s \geq 0} \binom{n}{m-s} \binom{p}{s}.$$

By [8] chapter 4.5, formulae (21), for fixed n and m , takes place the following relation

$$\sum_{m_1, m_2, \dots} \binom{m}{m_1, m_2, \dots, m_n} = \binom{n-1}{m-1},$$

where summation is over m_1, m_2, \dots, m_n , such that $m = m_1 + m_2 + \dots + m_n$, $n = m_1 + 2m_2 + \dots + nm_n$.

By these relations,

$$\begin{aligned} & \sum_{|\lambda|=n-1} \binom{m(\lambda)}{m_1(\lambda), m_2(\lambda), \dots} \binom{n}{m(\lambda)} \\ &= \sum_{s \geq 1} \binom{n}{s} \sum_{|\lambda|=n-1, m(\lambda)=s} \binom{s}{m_1(\lambda), m_2(\lambda), \dots} \\ &= \sum_{s \geq 1} \binom{n}{s} \binom{n-2}{s-1} = \sum_{s \geq 1} \binom{n}{s} \binom{n-2}{n-s-1} \\ &= \binom{2(n-1)}{n-1}. \end{aligned}$$

Lemma 3.2.

$$\lim_{n \rightarrow \infty} \binom{2n-2}{n-1}^{1/n} = 4.$$

Proof. For $a \in \mathbf{Z}$ denote by $a!!$ a product of positive integers $a, a-2, a-4$, and so on. For example, $(2n-1)!!$ is a product of odd numbers between 1 and $2n-1$. We have

$$(2n-4)!! \leq (2n-3)!! \leq (2n-2)!!.$$

Thus

$$\frac{2^{n-2}}{n-1} \leq \frac{(2n-3)!!}{(n-1)!} \leq 2^{n-1}.$$

Since

$$\binom{2(n-1)}{n-1} = \frac{2(n-1)!}{((n-1)!)^2} = \frac{2^{n-1}(2n-3)!!}{(n-1)!},$$

we have

$$\frac{2^{2n-3}}{n-1} \leq \binom{2(n-1)}{n-1} \leq 2^{2(n-1)}.$$

It remains to note that

$$\lim_{n \rightarrow \infty} \left(\frac{2^{2n-3}}{n-1} \right)^{1/n} = \lim_{n \rightarrow \infty} (2^{4n-2})^{1/n} = 4.$$

Proof of Theorem 1.1.

As we mentioned above any polylinear base element of free Novikov algebra of degree n corresponds to Young diagram of degree $n-1$. Suppose that it has all together m rows, namely m_1 rows with i_1 boxes, m_2 rows with i_2 boxes, etc, m_k rows with i_k boxes, where $i_1 > i_2 > \dots > i_k$. So, $\sum_{s=1}^k i_s m_s = n-1$, and such Young diagram looks like

$$\begin{array}{c} \bullet \quad \dots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ m_1 \quad \{ \quad \begin{array}{c} \vdots \quad \dots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \bullet \quad \dots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \bullet \quad \dots \quad \bullet \quad \bullet \end{array} \\ m_2 \quad \{ \quad \begin{array}{c} \vdots \quad \dots \quad \vdots \quad \vdots \\ \bullet \quad \dots \quad \bullet \quad \bullet \\ \vdots \quad \dots \quad \vdots \\ \bullet \quad \dots \quad \bullet \end{array} \\ m_k \quad \{ \quad \begin{array}{c} \vdots \quad \dots \quad \vdots \\ \bullet \quad \dots \quad \bullet \end{array} \end{array}$$

Set $m_i = 0$, if $i > k$. The Novikov diagram corresponding to such Young diagram, filled by n different letters, is uniquely defined by its first column. The first column can be choosen in

$$\binom{n}{m_1, m_2, \dots, m_n} = \binom{m}{m_1, m_2, \dots, m_n} \binom{n}{m}$$

ways. Therefore, by lemma 3.1

$$\dim R_n = \sum \binom{m}{m_1, m_2, \dots, m_n} \binom{n}{m} = \binom{2(n-1)}{n-1}$$

(summation is over m_1, m_2, \dots, m_n such that $\sum_s i_s m_s = n - 1$.) By Lemma 3.2, $Exp(Nov) = 4$. Other statements of theorem 1.1 is evident.

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